Computing maximal-exponent factors in an overlap-free word[☆]

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Abstract

The exponent of a word is the quotient of its length over its smallest period. The exponent and the period of a word can be computed in time proportional to the word length. We design an algorithm to compute the maximal exponent of all factors of an overlap-free word. Our algorithm runs in linear-time on a fixed-size alphabet, while a naive solution of the question would run in cubic time. The solution for non overlap-free words derives from algorithms to compute all maximal repetitions, also called runs, occurring in the word.

We also show there is a linear number of occurrences of maximal-exponent factors in an overlap-free word. Their maximal number lies between $0.66n$ and $2.25n$ in a word of length n. The algorithm can additionally locate all of them in linear time.

Keywords: Word, string, repetition, power, repeat, periodicity, word exponent, return word, algorithm, automaton.
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1. Introduction

We consider the question of computing the maximal exponent of factors (substrings) of a given word (string). The exponent of a word is the quotient of the word length over the word smallest period. For example `alfalfa` has period 3 and exponent 7/3, and `restore` has period 5 and exponent 7/5. A word with exponent e is also called an e-power. The exponent indicates better than the period the degree of repetitiveness of factors occurring in a word.

In this article we focus on factors whose exponent is at most 2. Such factors can uniquely be written as uvu where u is the longest border of uvu, that is, the longest proper prefix that is also a suffix of the factor. Note that the exponent is 1 if and only if u is the empty word, while it is 2 if and only if v is the empty word. Consistently with the existing literature a word whose exponent is 1, the minimal possible exponent, admits only the empty word as a border and is called border-free. A word is called a square when its exponent is a positive even integer. In this article, a factor whose exponent is smaller than 2 is called a repeat, while a factor whose exponent is at least 2 is called a repetition or a periodic factor. In other words, in the former case the factor u repeats at two distant positions.

The study of repeats in a word is relevant to long-distance interactions between separated occurrences of the same segment (the u part) in the word. Although occurrences may be far away from each other, they may interact when, for example, the word is folded as it is the case for genomic sequences. A very close problem to considering those repeats is that of computing maximal pairs (positions of the two occurrences of u) with gaps constraints as described by Gusfield [2] and later improved by Brodal et al. [3].

From a combinatorial point of view, the question is related to return words: z is a return word associated with u if u is a prefix of zu and u has no internal occurrence in zu. For instance, if u has only two occurrences in uvu (as a prefix and a suffix) then uv is a return word for u. The word then links two consecutive occurrences of u. The analysis of return words provide characterisations for word morphisms and infinite words. For example, a binary infinite Sturmian word, generalisation of Fibonacci word, is characterised by the fact that every factor (occurring infinitely many times) admits exactly two return words (see [4] and references therein).

The notion of maximal exponent is central to questions related to the avoidability of powers in infinite words. An infinite word is said to avoid

e-powers (resp. e^+-powers) if the exponents of its finite factors are smaller than e (resp. no more than e). Dejean [5] introduced the repetitive threshold $\mathrm{RT}(a)$ of an a-letter alphabet: the smallest rational number for which there exists an infinite word on a letters whose finite factors have exponent at most $\mathrm{RT}(a)$. In other words, the maximal exponent of factors of such a word is $\mathrm{RT}(a)$, the minimum possible. The word is also said to be $\mathrm{RT}(a)^+$-power free. It is known from Thue [6] that $r(2) = 2$, Dejean [5] proved that $r(3) = 7/4$ and stated the exact values of $\mathrm{RT}(a)$ for every alphabet size $a > 3$. Dejean's conjecture was eventually proved in 2009 after partial proofs given by several authors (see [7, 8] and references therein).

The exponent of a word can be calculated in linear time using basic string matching that computes the smallest period associated with the longest border of the word (see [9]). A straightforward consequence provides a $O(n^3)$-time solution to compute the maximal exponent of all factors of a word of length n since there are potentially of the order of n^2 factors. However, a quadratic time solution is also a simple application of basic string matching. By contrast, our solution runs in linear time on a fixed-size alphabet.

When a word contains runs, that is, maximal periodicities of exponent at least 2, computing their maximal exponent can be achieved in linear time by adapting the algorithm of Kolpakov and Kucherov [10] that computes all the runs occurring in the word. Their result relies on the fact that there exists a linear number of runs in a word [10] (see [11, 12] for precise bounds). Nevertheless, this does not apply to square-free words, which we are considering here.

Our solution works indeed on overlap-free words, not only on square-free words, that is, on words whose maximal exponent of factors is at most 2. Thus, we are looking for factors w of the form uvu, where u is the longest border of w. In order to accomplish this goal, we exploit two main tools: the Suffix Automaton of some factors and a specific factorisation of the whole word.

The Suffix Automaton (see [9]) is used to search for maximal-exponent factors in a product of two words due to its ability to locate occurrences of all factors of a pattern. Here, we enhance the automaton to report the rightmost occurrences of those factors. Exploiting only the Suffix Automaton in a balanced divide-and-conquer manner produces a $O(n \log n)$-time algorithm.

In order to eliminate the log factor we additionally exploit a word factorisation, namely the f-factorisation (see [9]), a type of LZ77 factorisation (see [13]) fit for word algorithms. It has now become common to use this factorisa-

tion to derive efficient or even optimal algorithms. The f-factorisation allows one to skip larger and larger parts of the words during an online processing. For our purpose, it is composed of factors occurring before their current position with no overlap. The factorisation can be computed in $O(n \log a)$-time (where a is the alphabet size) using a Suffix Tree or a Suffix Automaton, and in linear time on an integer alphabet using a Suffix Array [14].

The running time of the proposed algorithm depends additionally on the repetitive threshold of the underlying alphabet size of the word. The threshold restricts the context of the search for a second occurrence of u associated with a factor uvu.

We show a very surprising property of factors whose exponent is maximal in an overlap-free word: there are no more than a linear number of occurrences of them, although the number of occurrences of maximal (i.e. non-extensible) factors can be quadratic.

We show a lower bound of $0.66n$ and an upper bound of $2.25n$ on their maximal number for a word of length n. They improve on the bounds given in a preliminary version [1] of the article. The lower bound is based on a result of Pansiot [15] on the repetitive threshold of four-letter alphabets.

As a consequence, the algorithm can be modified to output all occurrences of maximal-exponent factors of an overlap-free word in linear time.

The question would have a simple solution by computing MinGap on each internal node of the Suffix Tree of the input word, as is discussed in the conclusion. MinGap of a node is the smallest difference between the positions assigned to leaves of the subtree rooted at the node. Unfortunately, the best algorithms for MinGap computation, equivalent to MaxGap computation, run in time $O(n \log n)$ (see [16, 17, 18] and the discussion in [19]).

A remaining question to the present study is to unify the algorithmic approaches for locating runs in non overlap-free words and maximal-exponent factors in overlap-free words.

The plan of the article is as follows. After defining the problem in the next section we present the general scheme of the algorithm that relies on the f-factorisation of the input word in Section 3. The sub-function operating a Suffix Automaton is described in Section 4 and the complexity of the complete algorithm is studied in Section 5. In Section 6 we prove lower and upper bounds on the number of occurrences of maximal-exponent factors. A conclusion follows.

2. Maximal-exponent factors

We consider words (strings) on a finite alphabet A of size a. If x is a word of length $|x| = m$, $x[i]$ denotes its letter at position i, $0 \leq i < m$. A factor of x is of the form $x[i]x[i+1]\dots x[j]$ for two positions i and j and is denoted by $x[i\,..\,j]$ (it is the empty word if $j < i$). It is a prefix of x if $i = 0$ and a suffix of x if $j = m-1$.

The word x has period p, $0 < p \leq m$, if $x[i] = x[i+p]$ whenever both sides of the equality are defined, i.e. for $i = 0, \dots, m-p-1$. The period of x, $\mathrm{period}(x)$, is its smallest period and its exponent is $\exp(x) = m/\mathrm{period}(x)$. For example, $\exp(\texttt{restore}) = 7/5$, $\exp(\texttt{mama}) = 2$ and $\exp(\texttt{alfalfa}) = 7/3$. An overlap-free word contains no factor of exponent larger then 2, that is, no factor of the form $bwbwb$ for a letter b and a word w.

We consider a fixed overlap-free word y of length n and deal with its factors having the maximal exponent among all factor exponents. They are called **maximal-exponent factor** or MEF for short. They have exponent at most 2 since y is overlap-free.

A MEF w in y is of the form uvu, where u is its longest border (longest factor that is both a prefix and a suffix of w). Then $\mathrm{period}(w) = |uv|$ and $\exp(w) = |uvu|/|uv| = 1 + |u|/\mathrm{period}(w)$. By convention, in the following we allow a border-free factor to be considered as a MEF of exponent 1, though it contains no repeat in the common sense since the repeating element u is empty and it can appear only if no letter in y appears more than once, i.e. if its length is no more than the alphabet size.

First note that a MEF uvu contains only two occurrences of u since this would produce a factor with a larger exponent. Second, any occurrence of the MEF uvu is maximal in the sense that it cannot be extended with the same period. That is, the two occurrences of u are followed by two distinct letters and preceded by two distinct letters. These remarks are stated in Lemmas 3 and 2 respectively.

The maximality of occurrences of repetitions in non overlap-free words implies their linear number but unfortunately this property does not hold for MEF occurrences.

3. Computing the maximal exponent of factors

The core result of the article is an algorithm, MaxExpFac, that computes the maximal exponent of factors of the overlap-free word y. The algorithm has to look for factors of the form uvu, for two words u and v, u being

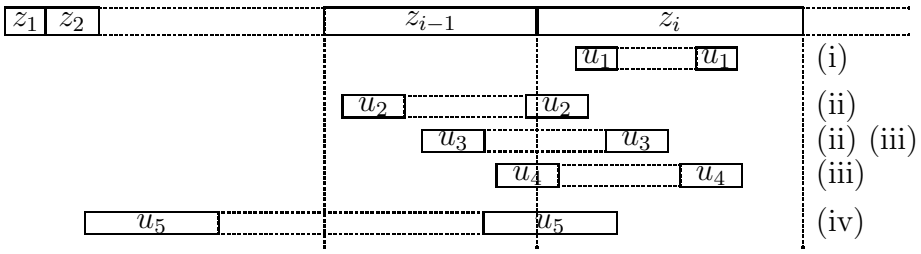

Figure 1: The only four possible locations of a factor uvu involving phrase z_i of the factorisation of the word: (i) internal to z_i; (ii) the first occurrence of u is internal to z_{i-1}; (iii) the second occurrence of u is internal to z_i; (iv) the second occurrence of u is internal to $z_{i-1}z_i$.

the longest border of uvu. The aim of this algorithm is accomplished with the help of Algorithm MaxExp, designed in the next section, which detects those factors occurring within the concatenation of two words.

Algorithm MaxExpFac relies on the f-factorisation of y, a type of LZ77 factorisation [13] defined as follows. It is a sequence of non-empty words, $z_1, z_2, \ldots, z_k$, called phrases and satisfying $y = z_1 z_2 \cdots z_k$ where z_i is the longest prefix of $z_i z_{i+1} \cdots z_k$ occurring in $z_1 z_2 \cdots z_{i-1}$. When this longest prefix is empty, z_i is the first letter of $z_i z_{i+1} \cdots z_k$, thus it is a letter that does not occur previously in y. This definition is equivalent to the definition in [9], in which a phrase z_i can overlap with its previous occurrence, because the word y is overlap-free. We adapt the factorisation to the purpose of our problem by defining z_1 as the longest prefix of y in which no letter occurs more than once. Then, $|z_1| \leq a$ and MaxExpFac$(z_1) = 1$. Note that MaxExpFac$(z_1 z_2) > 1$ if $z_1 \neq y$.

When the factorisation of y is computed, Algorithm MaxExpFac processes the phrases sequentially, from z_2 to z_k. After $z_1, z_2, \ldots, z_{i-1}$ have been processed, the variable e stores the maximal exponent of factors of $z_1 z_2 \cdots z_{i-1}$. Then, the next factors to be considered are those involving phrase z_i. Such factors uvu can either be internal to z_i or involve other phrases. However, the crucial property of the factorisation is that the second occurrence of u is only to be searched for in $z_{i-1}z_i$ because it cannot contain a phrase as this would contradict the definition of the factorisation.

We further distinguish four possible cases according to the position of the factor uvu as follows (see Figure 1):

(i) The two occurrences of u are contained in z_i.

(ii) The first occurrence of u is contained in z_{i-1} and the second ends in z_i.

(iii) The first occurrence of u starts in z_{i-1} and the second occurrence is contained in z_i.

(iv) The first occurrence of u starts in $z_1 \cdots z_{i-2}$ and the second occurrence is contained in $z_{i-1}z_i$.

Case (i) needs no action and other cases are dealt with calls to Algorithm MaxExp as described in the code below where $\widetilde{x}$ denotes the reverse of word x. For any two words z and w and a positive rational number e, MaxExp(z, w, e) is the maximal exponent of factors in zw whose occurrences start in z and end in w, and whose exponent is at least e; it is e itself if there is no such factor.

```
MaxExpFac(y)
 1  (z_1, z_2, ..., z_k) ← f-factorisation of y
 2  ▷ z_1 is the longest prefix of y in which no letter repeats
 3  e ← 1
 4  for i ← 2 to k do
 5       e ← MaxExp(z_{i-1}, z_i, e)
 6       e ← MaxExp(z̃_i, z̃_{i-1}, e)
 7       if i > 2 then
 8            e ← MaxExp(z_{i-1}z̃_i, z_1 ··· z_{i-2}, e)
 9  return e
```

Note that variable e can be initialised to the repetitive threshold $\mathrm{RT}(a)$ (see Introduction) when the alphabet of word y is of size a and if the word is long enough. The maximal length of words containing no factor of exponent at least $\mathrm{RT}(a)$ is 3 for $a = 2$, 38 for $a = 3$, 121 for $a = 4$, and $a + 1$ for $a \geq 5$ (see [5]).

Another technical remark is that the instruction at line 6 can be tuned to deal only with type (iii) factors of the form u_4vu_4 (see Figure 1), i.e. factors for which the first occurrence of the border starts in z_{i-1} and ends in z_i, because line 5 finds those of the form u_3vu_3. However, this has no influence on the asymptotic running time.

Theorem 1. *For any overlap-free word input,* MaxExpFac *computes the maximal exponent of factors occurring in the word.*

Proof. We consider a run of MaxExpFac(y). Let $e_1, e_2, \ldots, e_k$ be the successive values of the variable e, where e_i is the value of e just after the

execution of lines 5–8 for index i. The initial value $e_1 = 1$ is the maximal exponent of factors in z_1 as a consequence of its definition. We show that e_i is the maximal exponent of factors occurring in $z_1 z_2 \cdots z_i$ if e_{i-1} is that of $z_1 z_2 \cdots z_{i-1}$, for $2 \leq i \leq k$.

To do so, since e_i is at least e_{i-1} (use of max at lines 5–8), all factors occurring in $z_1 z_2 \cdots z_{i-1}$ are taken into account and we only have to consider factors coming from the concatenation of $z_1 z_2 \cdots z_{i-1}$ with z_i, that is, factors of the form uvu where the second occurrence of u ends in z_i. As discussed above and illustrated in Figure 1, only four cases are to be considered because the second occurrence of u cannot start in $z_1 z_2 \cdots z_{i-2}$ without contradicting the definition of z_{i-1}.

Line 5 deals with Case (ii) by the definition of MaxExp. Similarly, line 6 is for Case (iii), and line 8 for Case (iv).

If a factor occurs entirely in z_i, Case (i), by the definition of z_i it occurs also in $z_1 z_2 \cdots z_{i-1}$, which is reported by e_{i-1}.

Therefore, all relevant factors are considered in the computation of e_i, which is then the maximal exponent of factors occurring in $z_1 z_2 \cdots z_i$. This implies that the exponent e_k returned by the algorithm is the exponent of $z_1 z_2 \cdots z_k = y$ as stated. ∎

4. Locating repeats in a product

In this section, we describe Algorithm MaxExp applied to (z, w, e) for computing the maximal exponent of factors in zw that end in w, whose left border occurs in z, and whose exponent is at least e. MaxExp is called in the main algorithm of the previous section.

To locate factors under consideration, the algorithm examines positions j on w and for each computes the longest potential border of a factor, a longest suffix u of $zw[0\,..\,j]$ occurring in z. The algorithm is built upon an algorithm that finds all of them using the Suffix Automaton of word z.

The Suffix Automaton of z, denoted $\mathcal{S}(z)$, is used to locate borders of factors. It is the minimal deterministic finite automaton whose language is the set of suffixes of z (see [9, Section 6.6] for more description and for efficient construction). An example is given in Figure 2. The data structure has an initial state denoted initial($\mathcal{S}$) and a state called last($\mathcal{S}$) that is the accepting state of z itself (it is the only state with no outgoing arcs). In addition to the transition function goto (represented by arcs in the figure) it

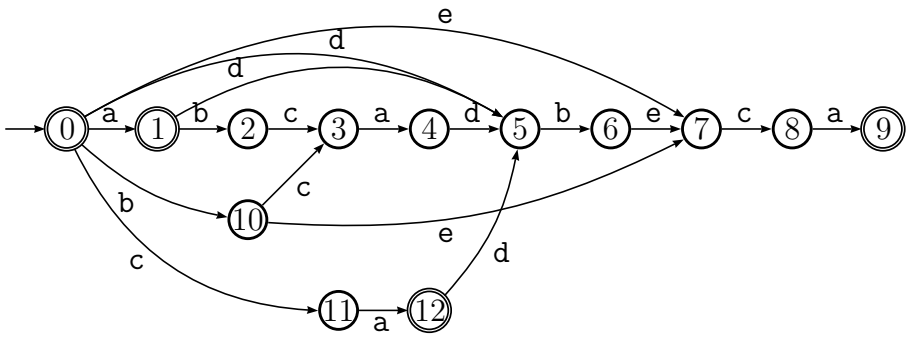

Figure 2: Suffix Automaton of `abcadbeca`. Suffix links: $F[1] = 0$, $F[2] = 10$, $F[3] = 11$, $F[4] = 1$, $F[5] = 0$, $F[6] = 10$, $F[7] = 0$, $F[8] = 11$, $F[9] = 12$, $F[10] = 0$, $F[11] = 0$, $F[12] = 1$. Maximal incoming word lengths: $L[0] = 0$, $L[1] = 1$, $L[2] = 2$, $L[3] = 3$, $L[4] = 4$, $L[5] = 5$, $L[6] = 6$, $L[7] = 7$, $L[8] = 8$, $L[9] = 9$, $L[10] = 1$, $L[11] = 1$, $L[12] = 2$. Minimal extension lengths: $sc[0] = 0$, $sc[1] = 0$, $sc[2] = 7$, $sc[3] = 6$, $sc[4] = 5$, $sc[5] = 4$, $sc[6] = 3$, $sc[7] = 2$, $sc[8] = 1$, $sc[9] = 0$, $sc[10] = 3$, $sc[11] = 1$, $sc[12] = 0$.

contains the failure link F_z and the length function L_z, both defined on the set of states. The link is defined as follows: let $p = \text{goto}(\text{initial}(\mathcal{S}(z)), x)$ for $x \in A^+$; then $F_z(p) = \text{goto}(\text{initial}(\mathcal{S}(z)), x')$, where x' is the longest suffix of x for which this latter state is not p. As for the length function, $L_z(p)$ is the maximal length of words x for which $p = \text{goto}(\text{initial}(\mathcal{S}(z)), x)$.

The next two lemmas show that, after u is located with the Suffix Automaton, although some of its suffixes may have an exponent higher than e, we can discard many of them.

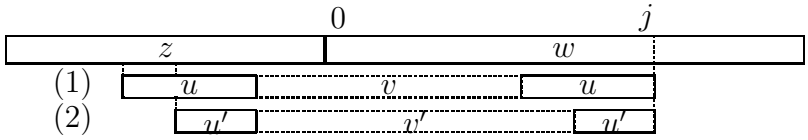

Figure 3: When u and its suffix u' end at the same right-most position on z, factor (1) has a larger exponent than factor (2).

Figure 3 illustrates the proof of the following lemma.

Lemma 2. *Let u' be a suffix of u. If they are both associated with the same state of $\mathcal{S}(z)$ the maximal exponent of a $u'v'u'$ is not greater than the maximal exponent of its associated uvu factor.*

Proof. The hypothesis implies that the right-most occurrence of u' ends at the same positions on z as u (see Figure 3). Then, $u'v'u'$ and uvu have

the same period $|vu| = |v'u'|$ but since $u'v'u'$ is not longer than uvu, the exponent of $u'v'u'$ is not greater than that of uvu. ∎

Note that a suffix u' of u may have an internal occurrence in uvu, which would lead to a factor having a larger exponent. For example, let $z = \texttt{abadba}$ and $w = \texttt{cdaba}$. The factor `abadbacdaba` with border `aba` has exponent 11/8 while the suffix `ba` of `aba` infers the factor `bacdaba` of greater exponent 7/5.

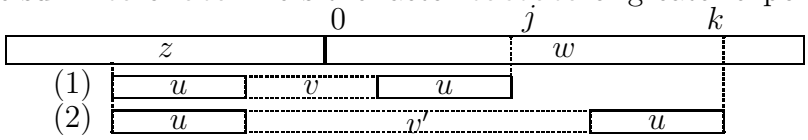

Figure 4: Factor (1) ending at position j has a larger exponent than factor (2) ending at position $k > j$.

The proof of the following lemma can be deduced from the remark in Figure 4.

Lemma 3. *If u occurs at end positions j and k on w with $k > j$, the factor $uv'u$ ending at k cannot be a maximal-exponent factor.*

Proof. To have a maximal exponent the first occurrence of u in $uv'u$ should end at the right-most position on z. But then there is a factor sharing the same first occurrence of u and with a closer second occurrence of u (see Figure 4). Therefore $1 + |u|/|uv| > 1 + |u|/|uv'|$, which completes the statement proof. ∎

The properties stated in the previous lemmas are used by Algorithm MaxExp to avoid some exponent calculations as follows. Let uvu be a factor ending at j on $zw[0\,..\,j]$ and for which u is the longest word associated with state $q = \text{goto}(\text{initial}(\mathcal{S}), u)$, where goto is the transition function of the automaton. Then next occurrences of u and of any of its suffixes cannot produce factors with an exponent larger than that of uvu. State q is then marked to inform the next steps of the algorithm that it has been visited.

We need another function, sc_z, defined on states of $\mathcal{S}(z)$ as follows: $sc_z(p)$ is the minimal length of paths from p to a terminal state; in other terms, if $p = \text{goto}(\text{initial}(\mathcal{S}(z)), x)$, then $sc_z(p) = |x'|$ where x' is the shortest word for which xx' is a suffix of z. With this precomputed extra element, computing an exponent is a mere division (see Figure 5).

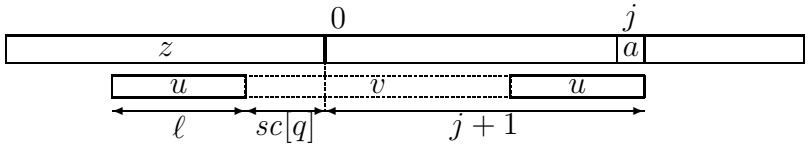

Figure 5: The maximal exponent of all factors in question bordered by u, longest factor of z ending at j, is $(\ell + sc[q] + j + 1)/(sc[q] + j + 1)$.

j		0	1	2	3	4	5	6	7	8	9
$w[j]$		d	e	c	a	d	b	e	c	a	d
q	12	5	7	8	9	5	6	7	8	9	5
ℓ	2	3	1	2	3	3	4	5	6	7	3
exp		8/5	5/4	3/2	7/4	4/3	13/9	14/9	5/3	16/9	17/14
				5/4			10/9				

Figure 6: Computing exponents when searching zw for factors uvu. The first occurrence of u is in z and the second ends in zw. The Suffix Automaton of $z =$ `abcadbeca` with function sc is in Figure 2. The search is done by parsing $w =$ `decadbecad` with the automaton. Exponents of factors are given by the expression $(\ell + sc[q] + j + 1)/(sc[q] + j + 1)$. The last line is for exponents corresponding to suffixes of u. The maximal exponent of all factors is 7/4.

```
MaxExp(z, w, e)
 1  S ← Suffix Automaton of z
 2  mark initial(S)
 3  (q, ℓ) ← (F[last(S)], L[F[last(S)]])
 4  for j ← 0 to min{⌊|z|/(e − 1) − 1⌋, |w| − 1} do
 5       while goto(q, w[j]) = NIL and q ≠ initial(S) do
 6            (q, ℓ) ← (F[q], L[F[q]])
 7       if goto(q, w[j]) ≠ NIL then
 8            (q, ℓ) ← (goto(q, w[j]), ℓ + 1)
 9            (q′, ℓ′) ← (q, ℓ)
10            while q′ unmarked do
11                 e ← max{e, (ℓ′ + sc[q′] + j + 1)/(sc[q′] + j + 1)}
12                 if ℓ′ = L[q′] then
13                      mark q′
14                 (q′, ℓ′) ← (F[q′], L[F[q′]])
15  return e
```

Figure 6 illustrates a computation done by the algorithm using the Suffix

Automaton of Figure 2.

Note that the potential overflow when computing $\lfloor |z|/(e-1)-1 \rfloor$ can easily be fixed in the algorithm implementation.

Theorem 4. *Algorithm* MaxExp*, applied to words z and w and to the rational number e, produces the maximal exponent of factors in zw that end in w, whose left border occurs in z, and whose exponent is at least e.*

Proof. In the algorithm, position j on w stands for a potential ending position of a relevant factor. First, we show that the algorithm does not require to examine more values of j than those specified at line 4. The exponent of a factor uvu is $|uvu|/|vu|$. Since we are looking for factors satisfying $|uvu|/|vu| \geq e$, the longest possible such factor has period $j+1$ and border z. Then $(|z|+j+1)/(j+1) > e$ implies $j < |z|/(e-1)-1$ (which is conventially set to $+\infty$ if $e = 1$). Since j is a position on w, $j < |w|$, which completes the first statement.

Second, given a position j on w, we show that the algorithm examines all the possible concerned factors having an exponent at least e and ending at j. The following property related to variables q, state of $\mathcal{S}$, and ℓ is known from [9, Section 6.6]: let u be the longest suffix of $zw[0\,..\,j]$ that is a factor of z, then $q = \text{goto}(\text{initial}(\mathcal{S}), u)$ and $\ell = |u|$. The property is also true just after execution of line 3 for z alone due to the initialisation of the two variables.

Then, word u is the border of a factor ending in w and whose left border occurs in z. Lines 9 to 14 check the exponents associated with u and its suffixes. If q' is unmarked, the exponent is computed as explained before (see Figure 5). If the condition at line 11 is met, which means that u is the longest word satisfying $q' = \text{goto}(\text{initial}(\mathcal{S}), u)$, due to Lemma 3 the algorithm does not need to check the exponent associated with later occurrences of u, nor with the suffixes of u since they have been checked before. Due to Lemma 2, suffixes of u ending at the same right-most position on z do not have a larger exponent. Therefore the next suffix whose associated exponent has to be checked is the longest suffix leading to a different state of $\mathcal{S}$: it is $F(q')$ and the length of the suffix is $L(F(q'))$ by definition of F and L.

Finally note the initial state of $\mathcal{S}$ is marked because it corresponds to an empty word u, that is a factor of exponent 1, which is not larger than the values of e.

This proves the algorithm runs through all possible relevant factors and completes the proof. ∎

5. Complexity analysis

In this section we analyse the running time and memory usage of our algorithms.

Proposition 5. *Applied to words z and w and to the rational number e, Algorithm* MaxExp *requires $O(|z|)$ space in addition to inputs and runs in total time $O(|z| + \min\{\lfloor |z|/(e-1) - 1 \rfloor, |w| - 1\})$ on a fixed size alphabet. It performs less than $2|z| + \min\{\lfloor |z|/(e-1) - 1 \rfloor, |w| - 1\}$ exponent computations.*

Proof. The space is used mostly for storing the automaton, which is known to have no more than $2|z|$ states and $3|z|$ edges (see [9]). It can be stored in linear space if edges are implemented by successor lists, which adds a multiplicative $\log a$ factor on transition time.

It is known from [9, Section 6.6] that the algorithm runs in linear time on a fixed alphabet, including the automaton construction with elements F, L and sc, if we exclude the time for executing lines 9 to 14.

It remains to enumerate the number of times line 11 is executed. It is done once for each position j associated with an unmarked state. If it is done more than once for a given position, then the second value of q' comes from the failure link. A crucial observation is that condition at line 12 holds for such a state. Therefore, since $\mathcal{S}(z)$ has no more than $2|z|$ states, the total number of extra executions of line 11 is at most $2|z|$, which gives the stated result. ∎

The proof of the linear running time of Algorithm MaxExpFac additionally relies on a combinatorial property of words. It is the notion of repetitive threshold $\mathrm{RT}(a)$ for an alphabet of size a mentioned in Introduction.

Theorem 6. *Applied to any overlap-free word of length n on a fixed-size alphabet, Algorithm* MaxExpFac *runs in time $O(n)$ and requires $O(n)$ extra space.*

Proof. Computing the f-factorisation $(z_1, z_2, \ldots, z_k)$ of the input takes time and space $O(n)$ on a fixed-size alphabet using any suffix data structure. (It can even be done in time $O(n)$ on an integer alphabet, see [14].)

The next instructions execute in linear extra space from Proposition 5. Line 5 takes time $O(|z_{i-1}| + \min\{\lfloor |z_{i-1}|/(e-1) - 1 \rfloor, |z_i| - 1\})$, which is bounded by $O(|z_{i-1}| + |z_{i-1}|/(e-1) - 1)$, for $i = 2, \ldots, k$. For a long enough

input, e is eventually at least $\mathrm{RT}(a)$ where a is the input alphabet. The time is then bounded by $O(|z_{i-1}| + |z_{i-1}|/(\mathrm{RT}(a) - 1) - 1)$, thus $O(|z_{i-1}|)$ because $\mathrm{RT}(a)$ is a constant. The contribution of Line 5 to the total runtime is then $O(\Sigma_{i=2}^{k}|z_{i-1}|)$.

Similarly it is $O(\Sigma_{i=2}^{k}|z_i|)$ for Line 6 and $O(\Sigma_{i=2}^{k}|z_{i-1}z_i|)$ for Line 8. Thus the overall runtime is bounded by $O(\Sigma_{i=1}^{k}|z_i|)$, which is $O(n)$ as expected. ■

6. Counting maximal-exponent factors

This section is devoted to the combinatorial aspects of maximal-exponent factors (MEF). We exhibit upper and lower bounds on their maximal number of occurrences in an overlap-free word.

The upper bound shows there is no more than a linear number of MEF occurrences in a word according to its length. In addition, the lower bound proves that this is optimal up to a multiplicative factor that remains to be discovered.

Note that on the alphabet $\{\mathtt{a}, \mathtt{a}_1, \ldots, \mathtt{a}_n\}$ the word $\mathtt{a}\mathtt{a}_1\mathtt{a}\mathtt{a}_2\mathtt{a}\ldots\mathtt{a}\mathtt{a}_n\mathtt{a}$ of length $2n+1$ has a quadratic number of maximal factors. Indeed all occurrences of factors of the form $\mathtt{a}w\mathtt{a}$ for a non-empty word w are non extensible. But only the n factors of the form $\mathtt{a}c\mathtt{a}$ for a letter c have the maximal exponent $3/2$.

6.1. Upper bound

Before giving an upper bound, we start with a simple property of MEFs, which does not lead to their linear number, but is used later to tune the upper bound.

Lemma 7. *Consider two occurrences of MEFs with the same border length b starting at respective i and j on the word y, $i < j$. Then, $j - i > b$.*

Proof. The two MEFs having the same border length, since they have the same exponent, they have also the same period and the same length. Let b be their border length and p their period.

Assume ab absurdo $j - i \leq b$. The word $y[i\,..\,i+b-1] = y[i+p\,..\,i+p+b-1]$ is the border of the first MEF. The assumption implies that $y[i+b] = y[i+p+b]$ because these letters belong to the border of the second MEF. It means the first MEF can be extended with the same period, producing a larger exponent, a contradiction. Therefore, $j - i > b$ as stated. ■

If we count the occurrences of MEFs by their border lengths after Lemma 7 we get an initial part of the harmonic series, a quantity that is not linear with respect to the length of y.

To get a linear upper bound on the number of occurrences of MEFs we introduce the notion of δ-MEFs, for a positive real number δ, as follows. A MEF uvu is a δ-MEF if its border length $b = |u| = |uvu| - \text{period}(uvu)$ satisfies $2\delta < b \leq 4\delta$. Then any MEF is a δ-MEF for some $\delta \in \Delta$, where $\Delta = \{1/4, 1/2, 1, 2, 2^2, 2^3, \ldots\}$. This is the technique used for example in [11, 12] to count runs in words.

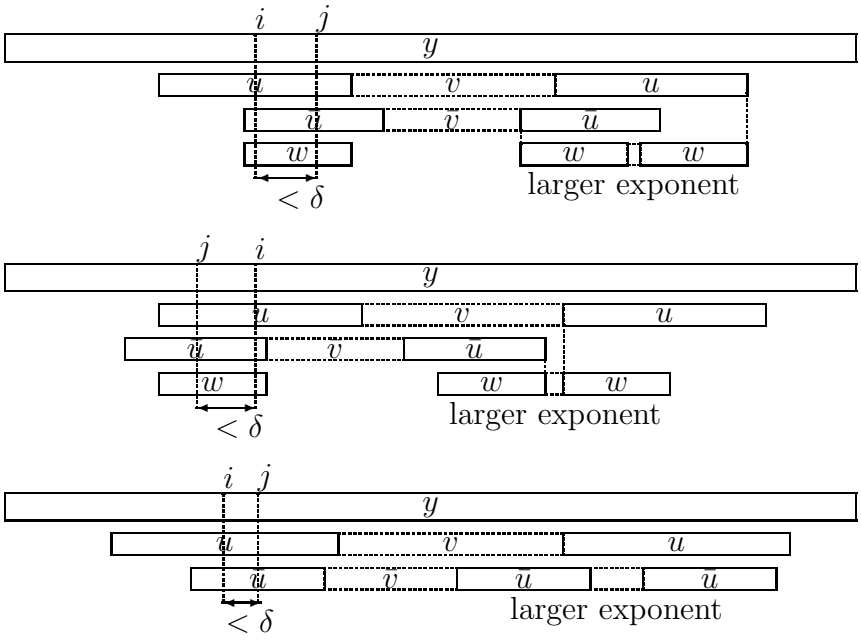

Figure 7: Two δ-MEFs, uvu and $\bar{u}\bar{v}\bar{u}$, having mid-positions of their left borders at close positions induce a factor with a larger exponent, a contradiction.

The proof of the next lemma is illustrated by Figure 7. We define the mid-position of an occurrence of a factor x whose first letter is at position i on y by $i + \lfloor |x|/2 \rfloor - 1$.

Lemma 8. *Let uvu and $\bar{u}\bar{v}\bar{u}$ be two occurrences of δ-MEFs in y whose left borders mid-positions are at respective positions i and j on y. Then, $|j-i| \geq \delta$.*

Proof. We consider w.l.o.g. $|u| \geq |\bar{u}|$. Assume ab absurdo $|j-i| < \delta$ (see Figure 7).

Since both $|u| > 2\delta$ and $|\bar{u}| > 2\delta$, the two occurrences of left borders overlap. Let w be the overlap. It can be a suffix of u and a prefix of $\bar{u}$, or it can be a suffix of $\bar{u}$ and a prefix of u, or w can be $\bar{u}$ itself, the shorter of two borders, when it occurs inside u. The three cases are displayed in this order on Figure 7.

Let $p = |uv|$ be the period of uvu and $p' = |\bar{u}\bar{v}|$ be that of $\bar{u}\bar{v}\bar{u}$. The exponent of the two factors is $e = 1 + |u|/p = 1 + |\bar{u}|/p'$, which implies $p - p' = (|u| - |\bar{u}|)/(e-1)$.

Note that w, the overlap of the two left borders, occurs at least at two other positions. For example, in the first case, it occurs as a suffix of the right border of u and as a prefix of the right border of $\bar{u}$. Due to the periodicity of the two factors, uvu and $\bar{u}\bar{v}\bar{u}$, the last two occurrences of w are $p - p'$ positions apart. Therefore the factor z starting with one occurrence and ending with the other has exponent at least (it can be larger if w is not the longest border of z):

$$1 + \frac{|w|}{p - p'} = 1 + \frac{|w|(e-1)}{(|u| - |\bar{u}|)}.$$

Now, from inequalities $2\delta < |\bar{u}| \leq |u| \leq 4\delta$ and the definition of w, we have both $|w| > |u|/2$ and $|u| - |\bar{u}| < |u|/2$. Then $|w| > |u| - |\bar{u}|$ and since $e - 1 > 0$ the exponent of z is then larger than e, a contradiction. Therefore $|j - i| \geq \delta$ as stated. ∎

A direct consequence of the previous lemma is the linear number of MEF occurrences. Because Lemma 8 implies that the number of δ-MEF occurrences in y is no more than n/δ. And since values of δ in Δ cover all border lengths, the total number of occurrences of MEFs is bounded by

$$\sum_{\delta \in \Delta} \frac{n}{\delta} = n \left(4 + 2 + 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots \right) < 8\, n.$$

The next statement refines the above upper bound by combining results of Lemmas 7 and 8.

Theorem 9. *There are less than* $2.25\,n$ *occurrences of maximal-exponent factors in a word of length* n*.*

Proof. According to Lemma 7 there are less than

$$\sum_{b=1}^{b=5} \frac{n}{b+1} = 1.45\,n$$

occurrences of MEFs with border length at most 5.

We then apply Lemma 8 with values of $\delta \in \Gamma$ that cover all remaining border lengths of MEFs: $\Gamma = \{(5/2), 5, 10, 20, \ldots\}$. It gives the upper bound

$$\sum_{\delta\in\Gamma} \frac{n}{\delta} = \frac{1}{5}\left(2 + 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots\right) n = \frac{4}{5}\,n$$

for the number of occurrences of MEFs with border length at least 6. Therefore the global upper bound we obtain is $2.25\,n$. ∎

Note that the border length 5 minimises the expression

$$\left(\sum_{b=1}^{b=k} \frac{n}{b+1}\right) + \frac{1}{k}\left(2 + 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots\right) n = \left(\sum_{b=1}^{b=k} \frac{n}{b+1}\right) + \frac{4\,n}{k}$$

with respect to k, which means the technique is unlikely to produce a smaller bound. By contrast, experiments show that the number of occurrences of MEFs is smaller than n and not even close to n, at least for small values of n. The following table displays the maximal number of MEFs for overlap-free word lengths $n = 5, 6, \ldots, 20$ and for alphabet sizes 2, 3 and 4. It also displays (second element of pairs) the associated maximal exponent. In the binary case we already know that it is 2 since squares are unavoidable in words whose length is greater than 3.

n	5	6	7	8	9	10	11	12
binary	2	3	4	5	5	6	6	8
ternary	(2, 1.5)	(3, 1.5)	(4, 2)	(5, 2)	(5, 2)	(6, 1.5)	(6, 2)	(8, 2)
4−ary	(2, 1.5)	(3, 1.5)	(4, 2)	(5, 2)	(5, 2)	(6, 1.5)	(7, 1.5)	(8, 2)

13	14	15	16	17	18	19	20
8	9	9	11	11	12	12	14
(8, 2)	(9, 2)	(9, 2)	(11, 2)	(11, 2)	(12, 2)	(12, 2)	(14, 2)
(8, 1.5)	(9, 1.5)	(10, 1.5)	(11, 2)	(12, 1.5)	(12, 1.5)	(13, 1.5)	(14, 1.5)

433 *6.2. Lower bound*

434 We now deal with a lower bound on the maximal number of occurrences
435 of maximal-exponent factors. We first consider an infinite word whose factors
436 have maximal exponent 3/2 and then show that its prefixes contain a linear
437 number of occurrences of these factors.

There exists an infinite word on the four-letter alphabet $A_4 = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$ whose maximal exponent of its factors is 7/5. The existence of such a word was proved by Pansiot [15] and it is easy to see that the exponent value cannot be smaller for an infinite word on A_4. Indeed, the result is part of the conjecture of Dejean [5] who stated the repetitive threshold for all alphabet sizes; the proof of this conjecture was eventually completed by Rao [7] and by Currie and Rampersad [8]. Here is an example of such a word given by Pansiot [15]:

$$\mathbf{p} = \mathtt{bacdabcadcbacdbcabdacbad}\ldots$$

From the word $\mathbf{p}$ we define $\mathbf{q}$ on the alphabet $A_5 = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ by inserting letter e in between any two consecutive letters. That is, for each integer $i \geq 0$,

$$\begin{aligned} \mathbf{q}[2i] &= \mathtt{e} \\ \mathbf{q}[2i+1] &= \mathbf{p}[i] \end{aligned}$$

or in other words $\mathbf{q} = f(\mathbf{p})$, where f is the morphism defined by $f(a) = \mathtt{e}a$, for any letter $a \in A_4$. The word $\mathbf{q}$ is:

$$\mathbf{q} = \mathtt{ebeaecedeaebeceaedecebeaecedebeceaebedeaecebeaed}\ldots$$

438 Let uvu be a factor of $\mathbf{p}$, where u is its longest border and then $|uv|$ is its
439 smallest period. By the choice of $\mathbf{p}$, we have $\exp(uvu) = |uvu|/|uv| \leq 7/5$.
440 In addition, we know that the period length of all 7/5-powers in $\mathbf{p}$ is at
441 least 10 (see [20]). Thus the induced factor $f(uvu)\mathtt{e}$ in $\mathbf{q}$ has exponent
442 $(2|uvu| + 1)/2|uv|$, which is 29/20 when uvu is a 7/5-power. This value is
443 less than 3/2.

444 As another example, consider the factor abcda of $\mathbf{p}$. It has exponent 5/4
445 and its induced factor in $\mathbf{q}$, $f(\mathtt{abcda})\mathtt{e} = \mathtt{eaebecedeae}$, has exponent 11/8,
446 which is less than 3/2 again. By contrast, the factor abca of $\mathbf{p}$ has exponent
447 4/3 and its induced factor in $\mathbf{q}$, eaebeceae has exponent $9/6 = 3/2$.

448 The next lemma shows that very few factors of $\mathbf{q}$ have exponent 3/2, the
449 maximal value.

Lemma 10. *Let w be a factor of $\mathbf{q}$, then* $\exp(w) \leq 3/2$. *Additionally* $\exp(w) = 3/2$ *when* $w = f(uvu)\mathtt{e}$ *with either* $uvu = v = \mathtt{a}$ *or* $u = \mathtt{a}$ *and* $v = \mathtt{bc}$ *up to a permutation of letters.*

Proof. Let w be a factor with maximal exponent among the factors of $\mathbf{q}$. Its first letter is $\mathtt{e}$ because otherwise its length could be increased by one unit without changing the period, which would increase the exponent. Similarly, its last letter is $\mathtt{e}$. Then, w is of the form $f(uvu)\mathtt{e}$ for a factor uvu of $\mathbf{p}$ whose longest border is u.

Assume that $\exp(w) \geq 3/2$. Then

$$\frac{2|uvu|+1}{2|uv|} \geq 3/2\,,$$

which gives

$$2|u|+1 \geq |uv|\,.$$

Also, since uvu is a factor of $\mathbf{p}$, it satisfies

$$|uvu|/|uv| \leq 7/5\,,$$

which implies

$$\frac{5}{2}|u| \leq |uv|\,.$$

Therefore

$$\frac{5}{2}|u| \leq 2|u|+1\,,$$

which is only possible for $|u| = 0, 1, \text{or } 2$.

If $|u| = 0$, $|v| = |uv| = 1$, and the induced factor in $\mathbf{q}$ is of the form $\mathtt{e}a\mathtt{e}$, for a letter $a \in A_4$, and has exponent $3/2$.

If $|u| = 1$, $|uv| = 3$, and then uvu is of the form $\mathtt{abca}$ up to a permutation of letters, inducing a factor of exponent $3/2$ in $\mathbf{q}$.

Finally, if $|u| = 2$, $|uv| = 5$ and $\exp(uvu) = 7/5$. But as recalled above, no factor of $\mathbf{p}$ with that exponent has period 5. This case is impossible, which concludes the proof. ∎

The conclusion of the previous lemma is that the maximal exponent of factors is $3/2$. The lower bound on the occurrence number of $3/2$-powers in $\mathbf{q}$ requires another property of $\mathbf{p}$, which is used in the proof of the following corollary.

Corollary 11. *The number of occurrences of maximal-exponent factors in prefixes of* $\mathbf{q}$ *tends to* $2n/3$ *with the prefix length* n*.*

Proof. From the previous lemma, maximal-exponent factors in $\mathbf{q}$ are induced by factors of the form `a` or `abca`, up to a permutation of the four letter of A_4, in $\mathbf{p}$.

It is clear from the definition of $\mathbf{q}$ that at every two of its positions occur one of the factors `eae`, `ebe`, `ece`, `ede`. Their occurrence number then tends to $n/2$.

Turning to the other factors of exponent $3/2$, it is known that the six factors of the form *abca* appear at every three positions in $\mathbf{p}$. Indeed, an occurrence of `abca`, can extend to `abcad` and `abcadb` but not to `abcadbc` whose suffix `bcadbc` has exponent $6/4 = 3/2 > 7/5$. Therefore, the induced factors of exponent $3/2$ occur at every six positions in $\mathbf{q}$, leading to a limit of $n/6$.

Summing up the two limits, the occurrence numbers of $3/2$-powers in prefixes of $\mathbf{q}$ tend to $n/2 + n/6 = 2n/3$ as stated. ■

7. Conclusion

The result of Section 6 implies that Algorithm MaxExpFac can be modified to output all the MEFs occurring in the input word in the same asymptotic time. Indeed, the only occurrences of MEFs that are skipped by the algorithm when computing the maximal exponent are those occurring inside a phrase of the f-factorisation (Case (i) of Section 3). However storing the previous occurrences of MEFs and listing them can be done in time proportional to their number, which does not affect the asymptotic running time of the algorithm and yields the next statement.

Corollary 12. *All the occurrences of maximal-exponent factors of a word can be listed in linear time with respect to its length.*

The present work triggers the study of a uniform solution to compute both repetitions (of exponent at least 2) and repeats. However, exponent 2 seems to reflect a transition phase in the combinatorics of these studied objects. For instance, the number of repetitions in a word can be of the order of $n \log n$ and the number of maximal periodicities (runs) is linear, while the number of maximal occurrences of factor *uvu* can be quadratic.

An interesting question is to select factors related to repeats that occur only a linear number of times or slightly more. An attempt has been achieved in [21] where it is shown that the number of maximal repetitions of any exponent more than $1 + \epsilon$ is bounded by $\frac{1}{\epsilon} n \ln n$. See also the discussions at the end of [10] and of [22].

Other interesting problems are the exact evaluation of the maximal number of occurrences of MEF and the calculation of the maximal number of (distinct) MEFs occurring in a word.

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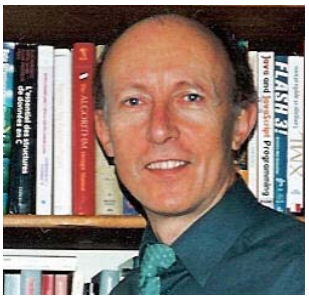

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